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ADDING AND SUBTRACTING JUMPS FROM MARKOV PROCESSES

BY

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ABSTRACT. If X_t is a continuous Markov process with infinitesimal generator A, if n is a kernel satisfying certain conditions, and if B is an operator given by

$$Bg(x) = \int [g(y) - g(x)]n(x, dy),$$

then $A + B$ will be the generator of a Markov process that has Lévy system (n, dt). Conversely, if X_t has Lévy system (n, dt), n satisfies certain conditions, and B is defined as above, then $A - B$ will be the generator of a continuous Markov process.

1. Introduction. Suppose X_t is a Markov process with infinitesimal generator A. If we perturb A by another operator B, will $A + B$ be the generator of a Markov process? And if so, what will the new Markov process look like? In this article we show that if X_t is continuous and B is given by $Bg(x) = \int[g(y) - g(x)]n(x, dy)$ for some kernel n, then $A + B$ will be the generator of a Markov process Y_t whose jump structure is completely described by n. Conversely, if A is the generator of a Markov process X_t whose jump structure is given by n and B is as given above, then $A - B$ will be the generator of a Markov process Y_t which has no jumps; that is, all the paths are continuous.

Previous work on this problem has been done by Cook **[4]**, in the case where $n(x, \cdot)$ is bounded in neighborhoods of x. There is a probabilistic construction of the new process Y_t due to Ikeda, Nagasawa, and Watanabe **[6]**, Meyer **[10]**, and Sawyer **[12]** (see §6) in the case $n(x, \cdot)$ is finite and sufficiently small. In this article, $n(x, \cdot)$ is allowed to be infinite (see Example 3.7). In probabilistic terms, $n(x, \cdot)$ being finite and sufficiently small means that for each path of X_t or Y_t, there will only be a finite number of jumps in each finite time interval; $n(x, \cdot)$ being infinite allows there to be infinitely many jumps in finite time intervals; in contrast to the n finite case, there may well be zero time between jumps.

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By X_t "has jump structure described by n", we mean X_t has Lévy system (n, dt), using the Lévy system developed by Watanabe [**14**] and Benveniste and Jacod [**2**]. Since the domains of infinitesimal generators are awkward to work with, we work instead with the resolvents R_λ and S_λ of X_t and Y_t, respectively. In §2, we give the necessary preliminaries. In §§3 and 4, we show that if BR_λ is bounded in norm, S_λ will be the resolvent of a semigroup in the cases where we are adding jumps and subtracting jumps, respectively. Example 3.7 is an example that shows n may be infinite. In §5, we show that Y_t has Lévy system (n, dt) or that Y_t is continuous depending whether we added or subtracted jumps. In §6 we describe the probabilistic construction of Y_t, when it exists.

§6 is almost entirely the work of P. A. Meyer and S. A. Sawyer, for which I gratefully thank them. I also would like to thank P. W. Millar for many helpful conversations.

2. Preliminaries. We will suppose E is a compact metric space. If E is only locally compact, we can make it compact by adding the point Δ, the one point compactification. $\| \ \|$ will denote the usual sup norm, both for functions and operators. $f_n \to f$ will mean $\|f_n - f\| \to 0$, unless specified otherwise. Let f_n converges weakly to f mean that $\sup\|f_n\| < \infty$ and $f_n(x) \to f(x)$ for all x. We will let $\mathcal{K}$ be the space of bounded, Borel measurable functions on E with sup norm, and $\mathcal{L}$ will refer to a closed subspace of $\mathcal{K}$. We will assume throughout that all kernels m satisfy $m(x, \{x\}) = 0$ for all x. Following the notation of [**3**], $X_t = (\Omega, \mathcal{F}_t, X_t, \theta_t, P^x)$ will be a right continuous strong Markov process with state space E.

A Lévy system (n, dH_t) for a process X_t is a kernel $n(x, dy)$ and a perfect continuous additive functional H_t such that for all $x \in E$, for all bounded stopping times T, and all positive Borel measurable functions f on $E \times E$ that are 0 on the diagonal, the Lévy system identity holds:

$$E^x \sum_{0<t<T} f(X_{t-}, X_t) = E^x \int_0^T \int f(X_t, y) n(X_t, dy)\, dH_t,$$

where both sides may be infinite.

Benveniste and Jacod [**2**] proved that every Hunt process has a Lévy system. We will assume throughout that $H_t(\omega) = t$ for all t and all ω. Since one can always perform a time change on X_t so that this is true (cf. [**9**, p. 150]), there is no real loss of generality. If X_t has Lévy system (n, dt), and $n'(x, dy) = n(x, dy)$ for all x except for a set of potential 0, it is clear that (n', dt) satisfies the Lévy system identity. We will sometimes refer to n as the Lévy kernel for X_t.

Given any kernel m, define the Lévy operator L_m associated with m by $L_m g(x) = \int [g(y) - g(x)] m(x, dy)$ for those g's and x's for which the integral is well defined. By the construction of Benveniste and Jacod, if m is a Lévy kernel for some Hunt process, $L_m g(x)$ is well defined for any g that vanishes in a neighborhood of x.

Let $m \geq n$ mean that except for a set of x's of potential 0, $m(x, F) \geq n(x, F)$ for all Borel sets F. m is bounded if $m(x, E)$ is a bounded function of x. Let us say that m_j increases strongly to m if each m_j is bounded, and there exist Borel sets F_j contained in $E \times E$ increasing to $E \times E$ such that $m_j(x, dy) = 1_{F_j}(x, y) m(x, dy)$. We will need

Proposition 2.1. *If m is the Lévy kernel of a Hunt process, there exist kernels m_j that increase strongly to m.*

Proof. If $F = F_1 \times F_2$, $1_{F_1 \times F_2}(x, y) m(x, dy)$ is clearly a kernel that is a measure in dy and is measurable in x. By a monotone class argument $1_F(x, y) m(x, dy)$ is a kernel for all F Borel in $E \times E$.

Let $G_i = \{(x, y): d(y, x) \geq 1/i\}$. Let $g_i(x) = \int 1_{G_i}(x, y) m(x, dy)$. Since m is a Lévy kernel, $g_i(x)$ is finite for all x. Also $g_i \uparrow$ as $i \to \infty$. Given a positive integer N, let $H_i = \{x: g_i(x) \leq N, g_k(x) > N \text{ for } k = i + 1, i + 2, \ldots\}$. Let $F'_N = \bigcup_{i=1}^{\infty} ((H_i \times E) \cap G_i)$. $1_{F'_N(x,y)} m(x, dy)$ is bounded by N. Now let $F_j = \bigcup_{N=1}^{j} F'_N$. Then $m_j(x, E)$ is bounded by $j(j + 1)/2$, where $m_j(x, dy) = 1_{F_j}(x, y) m(x, dy)$.

Note that if m_j increases strongly to m, $g \geq 0$ and $g(x) = 0$, $L_{m_j} g(x) = m_j(x, g) \to m(x, g) = L_m g(x)$.

Recall that if X_t is a Markov process, $P_t f(x) = E^x f(X_t)$ defines a semigroup P_t on $\mathcal{K}$. The resolvent R_λ of P_t is given by $R_\lambda f = \int_0^\infty e^{-\lambda t} P_t f\, dt$. If $\lambda \neq \mu$, R_λ satisfies the resolvent identity $R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu$.

Conversely, the Hille-Yosida theorem says that if $\{R_\lambda, \lambda > \lambda_0\}$ is a family of operators satisfying the resolvent identity, if $\|R_\lambda\| \leq 1/\lambda$, and if $R_\lambda(\mathcal{K})$ is dense (under the sup norm) in $\mathcal{L}$, a subspace of $\mathcal{K}$, then R_λ is the resolvent of a semigroup P_t that is strongly continuous on $\mathcal{L}$. An operator V is positive if $h \geq 0$ implies $Vh \geq 0$. By the construction of the Hille-Yosida theorem, if R_λ is positive, so is P_t.

If R_λ is the resolvent for a strong Markov process X_t, we have Dynkin's identity,

$$E^x R_\lambda g(X_T) - R_\lambda g(x) = E^x \int_0^T \lambda(\lambda R_\lambda - I) g(X_t)\, dt$$

for all $x \in E$, all $g \in \mathcal{K}$, and all bounded stopping times T, where I is the identity operator. We also know that if $g \in \mathcal{K}$, $R_\lambda g(X_t)$ is right continuous a.s.

If V is a positive linear operator and $V1 = c$, note that $\|V\| = c$, since if $-1 \leq g \leq 1, c - Vg = V(1 - g) \geq 0$ and $c + Vg = V(1 + g) \geq 0$.

If B is an operator such that $\|BR_\lambda\| < \delta < 1$ if $\lambda >$ some λ_0, we can define S_λ by $S_\lambda = R_\lambda(\sum_{i=0}^\infty (BR_\lambda)^i)$, and $\|S_\lambda\| \leq ((1-\delta)\lambda)^{-1}$.

$$S_\lambda = R_\lambda(I + BS_\lambda) = (I + R_\lambda B)S_\lambda.$$

We need a few basic properties of S_λ.

LEMMA 2.2. *If $R_\lambda = Q_\lambda(I + AR_\lambda)$ and $S_\lambda = R_\lambda(I + BS_\lambda)$, then $S_\lambda = Q_\lambda(I + (A + B)S_\lambda)$.*

PROOF. $S_\lambda = Q_\lambda(I + AR_\lambda)(I + BS_\lambda) = Q_\lambda(I + BS_\lambda) + Q_\lambda AR_\lambda(I + BS_\lambda) = Q_\lambda(I + BS_\lambda) + Q_\lambda AS_\lambda = Q_\lambda(I + (A + B)S_\lambda)$.

LEMMA 2.3. *If R_λ and S_λ each satisfy the resolvent identity for all μ and λ, and $S_\lambda = R_\lambda(I + BS_\lambda)$ for some λ, then $S_\mu = R_\mu(I + BS_\mu)$ for all $\mu > 0$.*

Proof is by direct computation using the resolvent identity for R_μ and S_μ.

3. Adding jumps. We assume throughout §3 that X_t is a strong Markov process with resolvent R_λ, n a kernel, B a linear operator such that $\|BR_\lambda\| < 1$ for all $\lambda >$ some λ_0. We let $S_\lambda = R_\lambda(\sum_{i=0}^\infty (BR_\lambda)^i)$, and our aim is to find conditions on B so that S_λ satisfies the conditions of the Hille-Yosida theorem. Note that $S_\lambda = R_\lambda(I + BS_\lambda) = (I + S_\lambda B)R_\lambda$ if $\lambda > \lambda_0$.

LEMMA 3.1. *S_λ satisfies the resolvent identity.*

The proof is identical to that of Lemma 2.1 of Leviatan [8], a fairly straightforward calculation.

LEMMA 3.2. *$S_\lambda(\mathcal{K})$ is dense in $\mathcal{L}$.*

PROOF. $R_\lambda(\mathcal{K})$ is dense in $\mathcal{L}$. If $g = R_\lambda f$, with $f \in \mathcal{K}$, let $h = (I - BR_\lambda)f$. Then $S_\lambda h = S_\lambda(I - BR_\lambda)f = R_\lambda f$.

We first prove our result for the case where the kernel n is bounded and B is the Lévy operator of n.

THEOREM 3.3. *Suppose $n(x, E) \leq N$ for all x and B is the Lévy operator of n. Then S_λ is the resolvent of a positive contraction semigroup on $\mathcal{L}$.*

PROOF. Let $\lambda \geq 4N$. Since $R_\lambda 1 = 1/\lambda$ and $B1 = 0$, it follows from Lemmas 3.1, 3.2 and §2 that we need only show that if $g \geq 0$ and $g \in \mathcal{L}$, then $S_\lambda g \geq 0$. We break the proof into two steps.

(1) Let $m = \inf_{x\in E} S_\lambda g(x)$. Let $\varepsilon > 0$. Since $S_\lambda g = R_\lambda(I + BS_\lambda)g$, $S_\lambda g(X_t)$ is right continuous. Let $T = \min(U, 1)$ where $U = \inf\{t > 0\colon\ S_\lambda g(X_t) - S_\lambda g(X_0) \geq 2\varepsilon\}$. We have $S_\lambda g(X_U) \geq S_\lambda g(X_0) + 2\varepsilon$ provided $U < \infty$. Suppose for the remainder of the proof that $S_\lambda g(x) < m + \varepsilon$.

If $P^x(T = U) > \frac{2}{3}$, then

$$\frac{\varepsilon}{3} \leq E^x S_\lambda g(X_T) - S_\lambda g(x)$$
$$= E^x \int_0^T (\lambda R_\lambda - I)(I + BS_\lambda) g(X_t)\, dt \leq 6\|g\| E^x T.$$

It follows that whether or not $P^x(T = U) > \frac{2}{3}$, we have $E^x T \geq \min(\frac{1}{3}, \varepsilon/(18\|g\|))$.

(2) If $S_\lambda g(y) < m + 2\varepsilon$, $-BS_\lambda g(y) \leq 2\varepsilon N$. We then have

$$m - S_\lambda g(x) \leq E^x S g(X_T) - S_\lambda g(x) = E^x \int_0^T (\lambda S_\lambda - I - BS_\lambda) g(X_t)\, dt$$
$$\leq E^x \int_0^T \lambda S_\lambda g(X_t)\, dt - E^x \int_0^T BS_\lambda g(X_t)\, dt$$
$$\leq \lambda(m + 2\varepsilon) E^x T + 2\varepsilon N E^x T.$$

Now if m were less than 0, we could pick ε small enough so that $\lambda(m + 2\varepsilon) + 2\varepsilon N < \lambda m/2$. We would then have that if $S_\lambda g(x) < m + \varepsilon$, $S_\lambda g(x) \geq m + \lambda(E^x T)|m|/2$, a contradiction to the definition of m since $E^x T$ is bounded away from 0 by step (1).

Comment. By the Hille-Yosida theorem there is a semigroup P_t such that

$$S_\lambda f = \int e^{-\lambda t} P_t f\, dt, \qquad \lambda \geq 4N, \quad f \in \mathfrak{L}.$$

We can use this equation to define $S_\lambda f$ for all λ. By Fubini's theorem and the fact that P_t is a semigroup, we have that S_λ satisfies the resolvent identity for all λ and $\mu > 0$. By Lemma 2.3, $S_\lambda = R_\lambda(I + BS_\lambda)$ for all λ, not just $\lambda \geq 4N$.

We now allow B to be unbounded.

THEOREM 3.4. *Suppose there exists a sequence of bounded kernels n_j, with Lévy operators L_{n_j}, and $L_{n_j} S_\lambda g \to BS_\lambda g$ for all $g \in \mathfrak{L}$. Then S_λ is the resolvent of a positive contraction semigroup on $\mathfrak{L}$.*

PROOF. Let $\lambda \geq \lambda_0$. Again, as in Theorem 3.3, we need only show $S_\lambda g \geq 0$ if $g \in \mathfrak{L}$ and $g \geq 0$. Let $S_j^\mu g = \sum_{i=0}^\infty R_\mu (L_{n_j} R_\mu)^i$ for $\mu \geq 2\|L_{n_j}\|$. By (3.3), S_j^μ is a resolvent for a semigroup, and $S_j^\mu = R_\mu + S_j^\mu L_{n_j} R_\mu$ for all μ. Since $S_\lambda = R_\lambda + R_\lambda BS_\lambda$,

$$S_j^\lambda(g - L_{n_j} S_\lambda g + BS_\lambda g) = R_\lambda g + R_\lambda BS_\lambda g$$
$$+ S_j^\lambda L_{n_j}(R_\lambda g + R_\lambda BS_\lambda g) - S_j^\lambda L_{n_j} S_\lambda g$$
$$= S_\lambda g.$$

Then $\|S_j^\lambda g - S_\lambda g\| \leq \|L_{n_j} S_\lambda g - BS_\lambda g\|/\lambda \to 0$ as $j \to \infty$. Since $S_j^\lambda g \geq 0$, our result follows.

COROLLARY 3.5. *Suppose B is the infinitesimal generator of a Markov process such that $BR_\lambda(\mathcal{L}) \subset \mathcal{L}$. Then S_λ is the resolvent of a contraction semigroup on $\mathcal{L}$.*

PROOF. First of all, it is clear that $BS_\lambda = \Sigma_{i=0}^{\infty}(BR_\lambda)^i$ maps $\mathcal{L}$ into $\mathcal{L}$. If T_λ is the resolvent of the Markov process generated by B, let $L_{n_j} = j(jT_j - I)$:

$$L_{n_j}g(x) = \int [\, g(y) - g(x)]j^2T_j(x, dy).$$

Since $L_{n_j}S_\lambda g = j(jT_j - I)S_\lambda g = jT_jBS_\lambda g \to BS_\lambda g$ as $j \to \infty$ if $g \in \mathcal{L}$, the result follows by Theorem 3.4.

Corollary 3.5 is also proved in [5] and [7]. As in the following example, Corollary 3.5 shows that in some cases perturbing by a drift term may be viewed as the limit of perturbation by jumps.

EXAMPLE 3.6. Suppose R_λ is the resolvent of Brownian motion on the real line, $B = d/dx$. Here $\mathcal{L}$ is $\mathcal{C}_0$, the continuous functions that vanish at infinity. If $g \in \mathcal{C}_0$, $R_\lambda g$ is twice differentiable, hence $BR_\lambda g$ is continuous, and it is easily checked that $BR_\lambda g \in \mathcal{C}_0$. Also, since

$$2\|g\| > \|\lambda R_\lambda g - g\| = \|AR_\lambda g\| = \left\|\tfrac{1}{2}(R_\lambda g)''\right\|,$$

where A is the generator of Brownian motion, the identity $\|f'\|^2 < 4\|f\|\,\|f''\|$ gives $\|BR_\lambda g\|^2 < 16\|R_\lambda g\|\,\|g\|$, or $\|BR_\lambda g\| < 4\|g\|/\lambda^{1/2}$. It follows by Corollary 3.5 that S_λ is the resolvent of a semigroup on $\mathcal{C}_0$; it is clear that the generator of this semigroup is $\frac{1}{2}d^2/dx^2 + d/dx$.

EXAMPLE 3.7. Suppose that R_λ is the resolvent of Brownian motion on the real line,

$$Bg(x) = \int [\, g(y) - g(x)]n(x, dy),$$

where $\int_{|y-x|<\varepsilon}|y - x|n(x, dy) \to 0$ uniformly in x as $\varepsilon \to 0$ where $N = \int \min[|y - x|, 1]n(x, dy)$ is bounded. Note that it is possible here for $n(x, E)$ to be infinite for all x. Define n_j by $n_j(x, dy) = 1_{D_j}(x, y)n(x, dy)$ where $D_j = \{(x, y)\colon d(x, y) > 1/j\}$. As in Proposition 2.1, n_j is a kernel bounded by jN. As in Example 3.6 $(R_\lambda g)'$ is bounded in norm if $g \in \mathcal{C}_0$, or $|R_\lambda g(y) - R_\lambda g(x)| < c|y - x|$, where $c = 4\|g\|/\lambda^{1/2}$. Since $R_\lambda g_i \to R_\lambda g$ weakly if $g_i \to g$ weakly, a monotone class argument gives the above inequality for all g, not just those in $\mathcal{C}_0$. Since $S_\lambda g = R_\lambda(I + BS_\lambda)g$, $|S_\lambda g(y) - S_\lambda g(x)| < c'|y - x|$, where $c' = 8\|g\|$, if λ is large enough.

$$\|BS_\lambda g - L_{n_j}S_\lambda g\| < \int_{|y-x|<1/j} |S_\lambda g(y) - S_\lambda g(x)|\, n(x, dy)$$

$$< 8\|g\| \int_{|y-x|<1/j} |y - x|n(x, dy) \to 0 \quad \text{as } j \to \infty.$$

A similar argument shows that $\|BR_\lambda\| \leq 2N\|g\|/\lambda + 4N\|g\|/\lambda^{1/2}$; hence $\|BS_\lambda\| \to 0$ as $\lambda \to \infty$, and clearly $\|BS_\lambda\| < 1$ if λ is large enough. It follows by Theorem 3.4 that S_λ is the resolvent of a contraction semigroup on $\mathcal{C}_0$. We will show in §5 that the associated Markov process has Lévy kernel n.

4. Subtracting jumps. We assume throughout §4 that X_t is a Hunt process with resolvent R_λ and Lévy system (n, dt), that B is a linear operator, and that $\|BR_\lambda\| < 1$ for all $\lambda >$ some λ_0. We let $S_\lambda = R_\lambda(\sum_{i=0}^\infty(-BR_\lambda)^i)$, and again our aim is to show that S_λ satisfies the conditions of the Hille-Yosida theorem.

THEOREM 4.1. *Suppose n is bounded by N and B is the Lévy operator of n. Then S_λ is the resolvent of a positive contraction semigroup on $\mathcal{L}$.*

PROOF. Let $\lambda \geq 4N$. As in Lemmas 3.1 and 3.2, S_λ satisfies the resolvent identity and $S_\lambda(\mathcal{K})$ is dense in $\mathcal{L}$. So we need only show that if $g \geq 0, g \in \mathcal{L}$, then $S_\lambda g \geq 0$. $S_\lambda g(X_t) = R_\lambda(I - BS_\lambda)g(X_t)$ is right continuous. Let $m = \inf_{x\in E} S_\lambda g(x)$. Let $\varepsilon > 0$. Let $T = \min(U, V, 1)$, where $U = \inf\{t > 0: S_\lambda g(X_t) - S_\lambda g(X_0) \geq 2\varepsilon\}$ and $V = \inf\{t > 0: X_t \neq X_{t-}\}$. As in Theorem 3.3, if $U < \infty$ we have $S_\lambda g(X_U) \geq S_\lambda g(X_0) + 2\varepsilon$. We break the remainder of the proof into two steps. Let us suppose throughout that $S_\lambda g(x) < m + \varepsilon$.

(1) First we show E^xT is bounded away from 0. We have that if $P^x(T = U) \geq \frac{7}{12}$,

$$\begin{aligned}\varepsilon/3 &\leq 2\varepsilon P^x(T = U) - 2\varepsilon P^x(T < U)\\ &\leq E^x(S_\lambda g(X_T) - S_\lambda g(x); T = U) + E^x(S_\lambda g(X_T) - S_\lambda g(x); T < U)\\ &= E^xR_\lambda(I - BS_\lambda)g(X_T) - R_\lambda(I - BS_\lambda)g(x)\\ &= E^x\int_0^T(\lambda R_\lambda - I)(I - BS_\lambda)g(X_s)\,ds \leq 6\|g\|E^xT.\end{aligned}$$

If $P^x(T = V) \geq \frac{1}{3}$, let $t_0 = 1/(6N)$. Then

$$P^x(V \leq t_0) \leq E^x\sum_{0<t\leq t_0} 1_{(X_t\neq X_{t-})} = E^x\int_0^{t_0} n(X_t, E)\,dt \leq Nt_0 = \frac{1}{6}.$$

In this case, $P^x(T \geq t_0) \geq \frac{1}{6}$, or $E^xT \geq 1/(36N)$.

Finally, if $P^x(T = U) < \frac{7}{12}$ and $P^x(T = V) < \frac{1}{3}$, we must have $P^x(T = 1) \geq \frac{1}{12}$. In any case, we have $E^xT \geq c = \min(\frac{1}{12}, 1/(36N)\varepsilon/(18\|g\|))$.

(2) We now show $m \geq 0$. Note

$$S_\lambda g(X_T) - S_\lambda g(x) - (S_\lambda g(X_T) - S_\lambda g(X_{T-})) \geq m - S_\lambda g(x).$$

If $t < T$, $S_\lambda g(X_t) = S_\lambda g(X_{t-})$, and so

$$S_\lambda g(X_T) - S_\lambda g(X_{T-}) = \sum_{0<t\leq T}(S_\lambda g(X_t) - S_\lambda g(X_{t-})).$$

Then by the Lévy system identity and Dynkin's identity,

$$m - S_\lambda g(x) \leq E^x S_\lambda g(X_T) - S_\lambda g(x) - E^x \sum_{0<t\leq T} (S_\lambda g(X_t) - S_\lambda g(X_{t-}))$$

$$= E^x \int_0^T (\lambda R_\lambda - I)(I - BS_\lambda) g(X_t)\, dt - E^x \int_0^T BS_\lambda g(X_t)\, dt$$

$$\leq E^x \int_0^T \lambda S_\lambda g(X_t)\, dt \leq \lambda(m + 2\varepsilon) E^x T.$$

But if m were less than 0, we could let $\varepsilon = |m|/4$, and by selecting x so that $S_\lambda g(x)$ were close enough to m, we would have a contradiction.

THEOREM 4.2. *Suppose there exists a sequence of bounded kernels n_j with Lévy operators L_{n_j} such that* (i) *n_j strongly increases to n and* (ii) *$L_{n_j} S_\lambda g \to BS_\lambda g$ for all $g \in \mathfrak{L}$. Then S_λ is the resolvent of a positive contraction semigroup on $\mathfrak{L}$.*

PROOF. Suppose $g \geq 0$, $g \in \mathfrak{L}$. Let $S_j^\mu g = \sum_{i=0}^\infty R_\mu(-L_{n_j} R_\mu)^i$ for $\mu > 2\|L_{n_j}\|$. The proof goes exactly as for Theorem 3.4 provided we show $S_j^\mu g \geq 0$. But the proof of Theorem 4.1 goes through exactly as before if we redefine V to be $\inf\{t > 0: (X_{t-}, X_t) \in F_j\}$ and we consider $\Sigma(S_\lambda g(X_t) - S_\lambda g(X_{t-}))1_{((X_{t-}, X_t) \in F_j)}$ instead of $\Sigma(S_\lambda g(X_t) - S_\lambda g(X_{t-}))$, where for each j, F_j is a set such that $n_j(x, dy) = 1_{F_j}(x, n) n(x, dy)$.

EXAMPLE 4.3. Consider Example 3.7. From the fact that if $S_\lambda = R_\lambda + S_\lambda BR_\lambda$, $R_\lambda = S_\lambda(I - BR_\lambda)$, it should be clear that if the process constructed in Example 3.7 does have Lévy system (n, dt), a consequence of §5, then the semigroup constructed as a result of Theorem 4.2 will just be that of Brownian motion.

5. Lévy system. At this point we make three assumptions. Suppose X_t is a Hunt process. We know by §§3 and 4 that S_λ is the resolvent of a semigroup Q_t on $\mathfrak{L}$.

I. *We assume Q_t can be extended to a semigroup Q_t on $\mathcal{K}$.*

Let Y_t be a Markov process that has Q_t as its semigroup.

II. *We assume there is a version of Y_t which is a Hunt process.*

By [2], Y_t has a Lévy system (m, dH_t).

III. *We assume that we can suppose that H_t is absolutely continuous with respect to Lebesgue measure for all ω.*

As a consequence of III, we may suppose that Y_t has a Lévy system of the form (m, dt) (see proof of Proposition 5.1 below).

I, II, and III can be shown to hold under fairly general conditions, in particular when B is bounded, but the proofs are long. See [1]. In some cases, however, it is easy to verify I, II, and III. For example, it is well known that if $\mathfrak{L}$ is the collection of bounded continuous functions, I and II must hold. A simple condition that guarantees III, much stronger than is necessary, is the following.

PROPOSITION 5.1. *Suppose that given any pair of disjoint compact sets* F_1 *and* F_2, *there exists an* $f \in \mathcal{K}$ (*depending on* F_1 *and* F_2) *such that* $R_\lambda f > 0$ *and* $R_\lambda f$ *is* 0 *on* F_1 *and bounded away from* 0 *on* F_2. *Then* III *holds.*

PROOF. If F_1 and F_2 are two disjoint compact sets, let $f \in \mathcal{K}$ such that $R_\lambda f > 0$ and $R_\lambda f$ is 0 on F_1 and bounded away from 0 on F_2, say by δ.

By Lemma 3.2 or its counterpart from the subtracting jumps case, $R_\lambda f = S_\lambda h$ for some $h \in \mathcal{K}$. Let $T = \inf\{t > 0\colon Y_{t-} \in F_1, Y_t \in F_2\}$. Let $U = \inf\{t > 0\colon Y_t \in F_1\}$. By Dynkin's identity,

$$\delta P^x(T \leq t) \leq E^x S_\lambda h(Y_{T\wedge t}) - E^x S_\lambda h(Y_{U\wedge t})$$
$$= E^x \int_{U\wedge t}^{T\wedge t} \lambda(\lambda S_\lambda - I)h(Y_s)\, ds \leq 2\lambda\|h\|t.$$

Let $T_0 = 0$, $T_{i+1} = T_i + T \circ \theta_{T_i}$, the $(i+1)$st time Y_t jumps from F_1 to F_2. An induction argument and the strong Markov property gives $P^x(T_i \leq t) \leq (ct)^i$, where $c = 2\lambda\|h\|/\delta$. Hence if $t \leq 1/(2c)$,

$$E^x \sum_{s \leq t} 1_{(Y_{s-} \in F_1, Y_s \in F_2)} \leq \sum_{i=0}^{\infty} iP^x(T_i \leq t) \leq 4ct.$$

Repeating the construction of Benveniste and Jacod, one can see that one can take their additive functional H_t so that $E^x H_t \leq c't$ for some constant c'. By [**11**, Lemma 6.6], $H_t \leq c't$ a.s.; hence $H_t = \int_0^t h(Y_u)\, du$ for some h, bounded by c' by Proposition 5.3 below. If we let $m'(x, dy) = h(x)m(x, dy)$, it follows that (m', dt) is a Lévy system for Y_t.

Since it should be clear from the context which we mean, we are using P^x, E^x to refer to probabilities and expectations for both X_t and Y_t.

Note that in Example 3.7, $R_\lambda(K)$ contains the collection of twice continuously differentiable functions with compact support, and hence satisfies Proposition 5.1.

We now want to show that if X_t is continuous and we added jumps, Y_t has Lévy system (n, dt) and that if X_t has Lévy system (n, dt) and we subtracted jumps, Y_t is continuous. We first prove some lemmas. Let us say $L_n \ll L_m$, where L_n and L_m are Lévy operators if: whenever G is an open set, g a continuous bounded function ≥ 0 with support contained in G^c, $1_G(x)L_n g(x) \leq 1_G(x)L_m g(x)$ for all x except possibly for a set of potential 0.

LEMMA 5.2. *A set* D *has potential* 0 *with respect to* X_t *if and only if it has potential* 0 *with respect to* Y_t.

PROOF. We have $R_\lambda 1_D = 0$. Hence $S_\lambda 1_D = R_\lambda 1_D \pm S_\lambda B R_\lambda 1_D = 0$. The "if" part follows by symmetry.

Proposition 5.3. *If Z_t is any Hunt process, $g \geq 0$, either $f \geq 0$ or f bounded, and*

$$E^x \int_0^t f(Z_u)\, du \leq E^x \int_0^t g(Z_u)\, du < \infty$$

for all x and t, then $f \leq g$ except for a set of Z-potential 0.

Proof. By Proposition 2.8 of **[11]**,

$$E^x \int_T^U f(Z_u)\, du \leq E^x \int_T^U g(Z_u)\, du$$

for all bounded stopping times $T \leq U$. Let $\varepsilon > 0$, K any compact contained in $\{x\colon f(x) > g(x) + \varepsilon\}$, F_i compact sets increasing to K^c such that F_i is disjoint from K. Let

$$T_{i1} = \inf\{t\colon Z_t \in K\}, \qquad U_{i1} = \inf\{t > T_{i1}\colon Z_t \in F_i\},$$
$$T_{i2} = \inf\{t > U_{i1}\colon Z_t \in K\}, \qquad U_{i2} = \inf\{t > T_{i2}\colon Z_t \in F_i\},$$

etc. By quasi-left continuity, $T_{ij} \to \infty$, $U_{ij} \to \infty$ as $j \to \infty$. If L is any real,

$$\sum_{j=1}^{\infty} E^x \int_{T_{ij}\wedge L}^{U_{ij}\wedge L} f(Z_u)\, du = E^x \int_0^L f(Z_u) 1_{(U \in [T_{ij}, U_{ij}] \text{ for some } j)}\, du$$

$$\to E^x \int_0^L f(Z_u) 1_K(Z_u)$$

as $i \to \infty$ by dominated convergence.

Combining with a similar equation for g, we have

$$E^x \int_0^L f(Z_u) 1_K(Z_u)\, du \leq E^x \int_0^L g(Z_u) 1_K(Z_u)\, du.$$

We must have that K has potential 0, and since L was arbitrary, the result follows.

Lemma 5.4. *Suppose Z_t is a Hunt process with Lévy system (m, dt), Lévy operator L_m, and resolvent V_λ. Suppose G is an open set, g is a nonnegative, bounded continuous function with support in G^c. Suppose $\lambda V_\lambda g \to g$ weakly as $\lambda \to \infty$. Suppose $\{g_\lambda\}$ is a collection of functions that converge weakly to g as $\lambda \to \infty$ and that D is a bounded operator such that $Dg_\lambda \to Dg$ weakly. Suppose T and U are two bounded stopping times such that $T(\omega) \leq t < U(\omega)$ implies $Z_t(\omega) \in G$. Then*

$$E^x \int_T^U \lambda^2 V_\lambda g(Z_t)\, dt \to E^x \int_T^U L_m g(Z_t)\, dt < \infty; \tag{1}$$

$$E^x \int_T^U \lambda V_\lambda D g_\lambda(Z_t)\, dt \to E^x \int_T^U Dg(Z_t)\, dt. \tag{2}$$

PROOF. (1) By the hypotheses on g, T, and U,

$$E^x\int_T^U \lambda^2 V_\lambda g(Z_t)\,dt = E^x\int_T^U \lambda(\lambda V_\lambda - I)g(Z_t)\,dt$$
$$= E^x\lambda V_\lambda g(Z_U) - E^x\lambda V_\lambda g(Z_T)$$
$$\to E^x g(Z_U) - E^x g(Z_T) = E^x \sum_{T<t<U}[g(Z_t) - g(Z_{t-})]$$
$$= E^x\int_T^U L_m g(Z_t)\,dt.$$

(2)

$$E^x\int_U^T \lambda V_\lambda D g_\lambda(Z_t)\,dt = E^x V_\lambda D g_\lambda(Z_u) - E^x V_\lambda D g_\lambda(Z_t) + E^x\int_T^U D g_\lambda(Z_t)\,dt$$
$$\to E^x\int_T^U D g(Z_t)\,dt.$$

LEMMA 5.5. *Suppose Z_t is a Hunt process with resolvent V_λ, Lévy system (m, dt) and Lévy operator L_m. Let D be a bounded operator such that whenever $\{f_n\}$ converges weakly to f, Df_n converges weakly to Df. Suppose that W_λ is a collection of positive linear operators such that $W_\lambda = V_\lambda(I - DW_\lambda)$, $\|\lambda W_\lambda\| \leq 1$, and $\lambda W_\lambda g$ converges weakly to g as $\lambda \to \infty$ whenever g is bounded and continuous. Then $L_m \gg D$.*

PROOF. Let G be open, g nonnegative, bounded, and continuous with support in G^c. Suppose T and U are stopping times such that if $T(\omega) \leq t < U(\omega)$, $Z_t(\omega) \in G$. Since $W_\lambda g \geq 0$,

$$E^x\int_T^U \lambda^2 V_\lambda g(Z_t)\,dt \geq E^x\int_T^U \lambda V_\lambda D(\lambda W_\lambda g)(Z_t)\,dt.$$

Letting $\lambda \to \infty$, by Lemma 5.4, we get

$$\infty > E^x\int_T^U L_m g(Z_t)\,dt \geq E^x\int_T^U D g(Z_t)\,dt.$$

Since $\|Dg\| < \infty$, the result now follows by an argument very similar to Lemma 5.3.

LEMMA 5.6. *Suppose m and n are two kernels, L_m and L_n their Lévy operators, and $L_m \gg L_n$. Then $m \geq n$.*

PROOF. Suppose $\{G_i\}$ is a countable open basis for the topology of E. For each G_i, select a countable dense subset $\{g_{ij}\}$ of the positive continuous functions with support in G_i^c. If

$$N = \{x\colon 1_{G_i}(x)L_m g_{ij}(x) < 1_{G_i}(x)L_n g_{ij}(x) \text{ for some } i, j\},$$

N has potential 0. Note that if $x \in G_i$, $L_m g_{ij}(x) = m(x, g_{ij})$ and similarly for

L_n. A monotone class argument shows that if F is any Borel set and $x \notin N$, $m(x, F \cap G_i^c) \geq n(x, F \cap G_i^c)$, for each i. It follows that $m \geq n$.

THEOREM 5.7. *Suppose X_t, R_λ, n, B, and S_λ are as in* §4. *Suppose B is the Lévy operator of n. Suppose n_j is a sequence of bounded kernels that increase strongly to n such that if L_{n_j} is the Lévy operator of n_j, $L_{n_j}R_\lambda g \to BR_\lambda g$ for all $g \in \mathcal{K}$. Then Y_t is continuous a.s.*

PROOF. Suppose Y_t has Lévy kernel $m > 0$ with Lévy operator L_m. Let m_k be bounded kernels strongly increasing to m, L_{m_k} the Lévy operators. If $V_\lambda = S_\lambda(\sum_{i=0}^\infty(-L_{m_k}S_\lambda)^i)$, V_λ is positive by Theorem 4.1 and the proof of Theorem 4.2. Now let $W_\lambda = V_\lambda(\sum_{i=0}^\infty((B - L_{n_j})V_\lambda)^i)$. Since $V_\lambda = S_\lambda(I - L_{m_k}V_\lambda)$, BV_λ will have norm < 1 if λ is large enough; hence for large enough λ so will $(B - L_{n_j})V_\lambda$. Since for all $f \in \mathcal{K}$, $W_\lambda f = R_\lambda h$ for some h (Lemma 2.2), $(L_{n_r} - L_{n_j})W_\lambda f \to (B - L_{n_j})W_\lambda f$. Hence by Theorem 3.4, W_λ is positive. By I and II (valid in this case by [1]), W_λ is the resolvent of a Hunt process, hence $\lambda W_\lambda g \to g$ weakly whenever g is continuous.

By Lemma 2.2, $W_\lambda = R_\lambda(I - (L_{n_j} + L_{m_k})W_\lambda)$. By Lemma 5.5, $B \gg L_{n_j} + L_{m_k}$. Since if $g(x) = 0$, $g \geq 0$, $L_{n_j}g(x) = n_j(x, g) \uparrow n(x, g) = Bg(x)$, letting $j \to \infty$ gives $B \gg B + L_{m_k}$. Similarly, letting $k \to \infty$, we get $0 \gg L_m$, or $m = 0$ by Lemma 5.6. Hence the expected number of jumps in finite time is 0 by the Lévy system identity, i.e., Y_t is continuous a.s.

Finally we show that if X_t is continuous and we added jumps, Y_t has Lévy system (n, dt).

THEOREM 5.8. *Suppose X_t, n, R_λ, B, and S_λ are as in* §3. *Suppose B is the Lévy operator for n. Suppose there exist bounded kernels n_j strongly increasing to n with Lévy operator L_{n_j} such that $L_{n_j}S_{\lambda_g} \to BS_\lambda g$ for all $g \in \mathcal{K}$, then Y_t has Lévy system (n, dt).*

Suppose Y_t has Lévy system m, Lévy operator L_m. We show $m \geq n$. Let

$$V_\lambda = R_\lambda\left(\sum_{i=0}^\infty ((B - L_{n_j})R_\lambda)^i\right).$$

As in Theorem 5.7, V_λ is positive. By Lemma 2.2, $V_\lambda = S_\lambda(I - L_{n_j}V_\lambda)$. Hence $L_m \gg L_{n_j}$. Since $L_{n_j}g(x) \uparrow Bg(x)$ if $g(x) = 0$, letting $j \to \infty$ gives $L_m \gg B$. By Lemma 5.6, $m \geq n$.

Next we must show $m \leq n$. Let m_k be bounded kernels strongly increasing to m, L_{m_k} the Lévy operators. Let j be fixed. Again letting $V_\lambda = R_\lambda(\sum_{i=0}^\infty((B - L_{n_j})R_\lambda)^i)$, we know V_λ is positive. Let p be the Lévy kernel for the process with resolvent V_λ, L_p the Lévy operator. We first show $L_p = L_m - L_{n_j}$.

If $T_\lambda = S_\lambda(\sum_{i=0}^\infty(-L_{m_k}S_\lambda)^i)$, T_λ is positive since Y_t has Lévy kernel $m \geq m_k$.

$$T_\lambda = R_\lambda\big(I + (B - L_{m_k})T_\lambda\big) = R_\lambda\big(I + \big((B - L_{n_j}) + (L_{n_j} - L_{m_k})\big)T_\lambda\big),$$

from which it follows that $T_\lambda = V_\lambda((I + L_{n_j} - L_{m_k})T_\lambda)$. By Lemma 5.5, $L_p \gg L_{m_k} - L_{n_j}$. Letting $k \to \infty$, $L_p \gg L_m - L_{n_j}$.

On the other hand, let p_r be a sequence of kernels increasing strongly to p, with Lévy operators L_{p_r}. If $U_\lambda = V_\lambda(\Sigma_{i=0}^\infty(-L_{p_r}V_\lambda)^i)$, U_λ is positive, and

$$U_\lambda = V_\lambda(I - L_{p_r}U_\lambda) = S_\lambda\big(I - (L_{p_r} + L_{n_j})U_\lambda\big).$$

Hence $L_m \gg L_{p_r} + L_{n_j}$. Letting $r \to \infty$, $L_p \ll L_m - L_{n_j}$. Hence $p = m - n_j$.

We know $n_j(x, dy) = 1_{H_j}(x, y)n(x, dy)$ for sets H_j, $m_k(x, dy) = 1_{F_k}(x, y)m(x, dy)$ for sets F_k. Let

$$q_{jk}(x, dy) = 1_{F_k}(x, y)m(x, dy) - 1_{F_k}(x, y)n_j(x, dy),$$

$s_{jk}(x, dy) = 1_{F_k \cap H_j^c}(x, y)n(x, dy)$, $L_{q_{jk}}$, $L_{s_{jk}}$ the associated Lévy operators. Since $n_j < n < m$, $q_{jk} > 0$. But $q_{jk} \leq m_k$, hence q_{jk} is bounded. $s_{jk}(x, dy) < 1_{F_k}(x, dy)m(x, dy)$, hence $s_{jk} < m_k$, and s_{jk} is also bounded. Note also that $(q_{jk} - s_{jk})(x, dy) = 1_{F_k}(x, y)(m - n)(x, dy)$, which is independent of j.

Since $q_{jk} < m - n_j$, the Lévy kernel for the process with resolvent V_λ,

$$W_\lambda = V_\lambda\left(\sum_{i=0}^{\infty}(-L_{q_{jk}}V_\lambda)^i\right)$$

is positive by §4. Then $Z_\lambda^j = W_\lambda(\Sigma_{i=0}^\infty(L_{s_{jk}}W_\lambda)^i)$ is positive by Theorem 3.4. If we denote $L_{q_{jk}} - L_{s_{jk}}$ simply by $L_{(k)}$, $Z_\lambda^j = V_\lambda(I - L_{(k)}Z_\lambda^j)$, hence

$$Z_\lambda^j = R_\lambda\big(I + (B - L_{n_j} - L_{(k)})Z_\lambda^j\big).$$

Letting $j \to \infty$, we get $Z_\lambda = R_\lambda(\Sigma_{i=0}^\infty(-L_{(k)}R_\lambda)^i)$ is positive. $Z_\lambda = R_\lambda(I - L_{(k)}Z)$. Since X_t is continuous, the Lévy operator of X_t is 0, and by Lemma 5.5, $0 \gg L_{(k)}$. Letting $k \to \infty$, by Lemma 5.6 we conclude that $m - n < 0$.

6. A probabilistic construction. It would be nice to have a probabilistic construction of Y_t. If $n(x, E)$ is finite and small enough, one exists. In general, as in Example 3.7, none is known.

Suppose X_t has finitely many jumps in finite time. Kill X_t at the time T of the first jump, then "restart" it with distribution X_{T-}. Proceed until the first jump of the new process and kill again, etc. "Restarting" can be made precise through the work of Ikeda, Nagasawa, and Watanabe **[6]** or Meyer **[10]**. See also Stroock [13] in the case n is bounded. That this pieced together process is the same as the process constructed by the methods of this paper follows readily from [12]. If X_t has infinitely many jumps in finite time, as when $n(x, E)$ is infinite, the time of the first jump will, in general, be identically equal to 0, and this method will fail.

To add jumps, kill the process according to the multiplicative functional $\exp(-n(X_t, E))$. At the time of death T, "restart" it with distribution $n(X_{T-}, dy)/n(X_T, E)$. Again it is not too difficult to check that this new process is the Y_t one would have constructed through the methods of this paper. Again, however, if $n(x, E)$ is infinite, this method will fail since $\exp(-n(X_t, E))$ would be identically 0.

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